IEEE TRANSACTIONS ON AUTOMATIC CONTROL VOL. AC-11, NO. 1 pp. 102 - 108 JANUARY, 1966

Semi-Analytic Approach to Root Locus

VENKATARAMA KRISHNAN

Semi-Analytic Approach to Root Locus

VENKATARAMA KRISHNAN

Abstract—The graphical method of constructing root locus is very well known. The analytical methods of obtaining root locus equations developed in the Soviet Union merit utilization due to the flexibility inherent in the manipulation of algebraic expressions. Arrays are developed whereby these algebraic expressions can be obtained from a knowledge of the open loop transfer function. Breakaway points on the real axis and intersection with the imaginary axis can be obtained directly from these arrays. Finally, arrays are formulated for the case where one of the parameters of the open loop transfer function varies.

I. Introduction

THE GRAPHICAL method of constructing the root locus of the closed loop poles, when the gain K is varied from zero to infinity, has been well developed [5], [11]. The analytic approach presented by Bendrikov and Teodorchik [1]–[4] of the U.S.S.R. seems to have some appeal in the U. S. as evidenced by the recent papers [8], [9]. Utilizing only analytic methods, a complete treatise on root locus equations of the fourth degree has been published very recently [12]. These two methods can be combined utilizing, on the one hand, the graphical representation of the root locus and, on the other, the simplifications that are possible with algebraic expressions.

The root locus of a closed loop transfer function $Y(s)$ is defined as the locus of the poles of $Y(s)$ as the gain K is varied from $-\infty$ *to* $+\infty$. In the case of a unity feedback system, it is given by the solution to the equation

$$G(s) = -\frac{1}{K} \tag{1}$$

where $G(s)$ is the open loop transfer function.

The locus of the closed loop poles as K is varied from $-\infty$ *to* 0 can be interpreted as the locus of the open loop poles (i.e., poles of $G(s)$) of the unity feedback system as K is varied from 0 to ∞, because of the following equations,

$$\left.\begin{aligned} KY(s) &= \frac{KG(s)}{1+KG(s)} \cdots \\ KG(s) &= \frac{KY(s)}{1-KY(s)} \end{aligned}\right\}. \tag{2}$$

Manuscript received June 10, 1965; revised October 18, 1965. This work was performed under the Joint Services Technical Advisory Committee Contract AF-49(638)-1402. The author acknowledges the facilities provided by the Polytechnic Institute of Brooklyn, N. Y.

The author is with the Dept. of Electrical Engineering, Polytechnic Institute of Brooklyn, N. Y.

This locus will be called the "complementary root locus" following the terminology proposed by Narendra [7] which is more appropriate than the "inverse root locus" proposed by Aseltine [6]. The advantages of the graphical representation of the root locus can be exploited fully if an algebraic expression can be easily formulated.

Simplifying operations on analytic expressions can be executed by a judicious shifting of the $j\omega$-axis of the s-plane, especially if there is an axis of vertical symmetry. (The σ-axis is the axis of horizontal symmetry for the root locus.) In the following sections, an algorithm is developed whereby the algebraic expression for the root locus can be written easily from the knowledge of the open loop transfer function $G(s)$, given as a ratio of polynomials $N(s)$ and $D(s)$.

II. The Root Locus Equations

The open loop transfer function $G(s)$ is expressed as a ratio of polynomials $N(s)$ and $D(s)$, of degrees m and n, respectively, where $m<n$.

$$G(s) = \frac{N(s)}{D(s)} = \frac{s^m + a_{m-1}s^{m-1} + a_{m-2}s^{m-2} + \cdots + a_0}{s^n + b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_0}\cdot \tag{3}$$

Expressing s as the complex frequency $(\sigma+j\omega)$, and separating the real and imaginary parts of $G(\sigma+j\omega)$, the equations governing the root locus are given by

$$\operatorname{Im} G(\sigma + j\omega) = 0 \tag{4}$$

$$\operatorname{Re} G(\sigma + j\omega) = -\frac{1}{K}\cdot \tag{5}$$

Using (3), (4) and (5) can be expressed in the form of a determinant

$$\begin{vmatrix} \operatorname{Re} N(\sigma+j\omega) & \operatorname{Im} N(\sigma+j\omega) \\ \operatorname{Re} D(\sigma+j\omega) & \operatorname{Im} D(\sigma+j\omega) \end{vmatrix} = 0 \tag{6}$$

$$\frac{\begin{vmatrix} \operatorname{Re} N(\sigma+j\omega) & \operatorname{Im} N(\sigma+j\omega) \\ -\operatorname{Im} D(\sigma+j\omega) & \operatorname{Re} D(\sigma+j\omega) \end{vmatrix}}{\begin{vmatrix} \operatorname{Re} D(\sigma+j\omega) & \operatorname{Im} D(\sigma+j\omega) \\ -\operatorname{Im} D(\sigma+j\omega) & \operatorname{Re} D(\sigma+j\omega) \end{vmatrix}} = -\frac{1}{K}\cdot \tag{7}$$

Equation (6) will be called the "root locus equation" and (7) will be called the "gain equation." If a pole of $G(s)$, $\sigma_1+j\omega_1$, satisfies (6), then the value of the gain corresponding to the pole can be obtained from (7). If the numerator $N(s)$ is a constant A then (6) and (7) reduce to

$$\operatorname{Im} D(\sigma + j\omega) = 0$$
$$\operatorname{Re} D(\sigma + j\omega) = -KA.$$

Since $N(s)$ and $D(s)$ are analytic in the s-plane, they can be expressed as a Taylor Series about any point $(\sigma, j\omega)$:

$$R_1(\sigma) - R_3(\sigma)\omega^2 + R_5(\sigma)\omega^4 \pm \cdots = 0. \qquad (13)$$

3) The last terms in (9) and (13) are given by

$(-1)^{(n+m-1)/2}\omega^{n+m-1}R_{n+m}(\sigma)$ if $n+m$ is odd, and

$(-1)^{(n+m-2)/2}\omega^{n+m-2}R_{n+m-1}(\sigma)$ if $n+m$ is even.

$$\left.\begin{aligned} N(\sigma + j\omega) &= \sum_{j=0}^{k}\left[(-1)^j \frac{\omega^{2j}}{(2j)!}\left(N^{(2j)}(\sigma) + j\omega\frac{N^{(2j+1)}(\sigma)}{2j+1}\right)\right] k = \begin{cases} \dfrac{m}{2} & m \text{ even} \\ \dfrac{m-1}{2} & m \text{ odd}\end{cases} \\ D(\sigma + j\omega) &= \sum_{j=0}^{k}\left[(-1)^j \frac{\omega^{2j}}{(2j)!}\left(D^{(2j)}(\sigma) + j\omega\frac{D^{(2j+1)}(\sigma)}{2j+1}\right)\right] k = \begin{cases} \dfrac{n}{2} & n \text{ even} \\ \dfrac{n-1}{2} & n \text{ odd}\end{cases} \end{aligned}\right\} \qquad (8)$$

where $N^{(2j)}(\ ) = 2j$th derivative with respect to σ.

Substituting (8) in (6) and (7), and after some simplifications the root locus equation can be expressed by

$$\omega[R_1(\sigma) - \omega^2 R_3(\sigma) + \omega^4 R_5(\sigma) \pm \cdots] = 0. \qquad (9)$$

The gain equation is given by

$$\frac{R_0(\sigma) - \omega^2 R_2(\sigma) + \omega^4 R_4(\sigma) - \omega^6 R_6(\sigma) \pm \cdots}{D_0(\sigma) - \omega^2 D_2(\sigma) + \omega^4 D_4(\sigma) - \omega^6 D_6(\sigma) \pm \cdots} = -\frac{1}{K}. \qquad (10)$$

The coefficients of the powers of ω^2, viz., $R_k(\ )$ and $D_k(\ )$ are given by

$$R_k(\sigma) = \sum_{r=0}^{k}(-1)^r \frac{N^{(r)}(\sigma)}{r!}\cdot\frac{D^{(k-r)}(\sigma)}{(k-r)!} \qquad (11)$$

$$D_k(\sigma) = \sum_{r=0}^{k}(-1)^r \frac{D^{(r)}(\sigma)}{r!}\,\frac{D^{(k-r)}(\sigma)}{(k-r)!}. \qquad (12)$$

4) The root locus equation represents an algebraic curve and is derived for a given ratio of polynomials *without any prior knowledge of the location of the poles and zeros.*

5) Terms of powers higher than ω^4 are present in (9) and (13), only if the sum of the degrees of the numerator and denominator of the open loop transfer function $G(s)$ exceeds 6. If the sum of the degrees of the numerator and denominator is less than 7, (13) involves solving a *quadratic* in ω^2 for any given σ.

6) The breakaway points on the real axis are the intersection of the off-the-real-axis locus and the real axis and hence are given by the real axis solutions to the equation

$$R_1(\sigma) = 0. \qquad (14)$$

III. Algorithm for the Root Locus and Gain Equations

Thecoefficients of the powers of ω^2 in the Taylor series expansion of $N(s)$ and $D(s)$ given by (8) are first arranged in an array called the *root locus array.*

$$\begin{array}{cccccccc} R_0(\sigma) & R_1(\sigma) & R_2(\sigma) & R_3(\sigma) & R_4(\sigma) & & R_m(\sigma) & R_n(\sigma) \\ N(\sigma) & -N^{(1)}(\sigma) & \dfrac{N^{(2)}(\sigma)}{2!} & -\dfrac{N^{(3)}(\sigma)}{3!} & \dfrac{N^{(4)}(\sigma)}{4!} & \cdots & (\pm 1)^m\dfrac{N^m(\sigma)}{m!} & 0\quad 0\quad 0 \\ D(\sigma) & D^{(1)}(\sigma) & \dfrac{D^{(2)}(\sigma)}{2!} & \dfrac{D^{(3)}(\sigma)}{3!} & \dfrac{D^{(4)}(\sigma)}{4!} & \cdots & \dfrac{D^{(m)}(\sigma)}{m!} & \cdots\dfrac{D^{(n)}(\sigma)}{n!} \end{array}.$$

The degrees of (11) and (12) are given by $(n+m-k)$ and $(2n-k)$, respectively. The following comments can be made with respect to the root locus (9):

1) The real axis $\omega = 0$ is always on the root locus.

2) The locus off the real axis is expressed by a *finite* series in ω^2:

In this array the odd order differentiations of the numerator $N(\sigma)$ are negative. The coefficients $R_0(\sigma)$, $R_1(\sigma)$, $R_2(\sigma)$, $R_3(\sigma)\cdots$, $R_k(\sigma)$ are obtained by taking the sum of the cross products, as indicated below, the subscript k being interpreted as the sum of $(k+1)$ cross products.

$$R_0(\sigma) = \begin{bmatrix} N(\sigma) \\ \\ D(\sigma) \end{bmatrix} = N(\sigma)D(\sigma) \tag{15}$$

$$R_1(\sigma) = \begin{bmatrix} N(\sigma) & -N^{(1)}(\sigma) \\ \\ D(\sigma) & D^{(1)}(\sigma) \end{bmatrix} = N(\sigma)D^{(1)}(\sigma) - D(\sigma)N^{(1)}(\sigma) \tag{16}$$

$$R_2(\sigma) = \begin{bmatrix} N(\sigma) & -N^{(1)}(\sigma) & \dfrac{N^{(2)}(\sigma)}{2!} \\ \\ D(\sigma) & D^{(1)}(\sigma) & \dfrac{D^{(2)}(\sigma)}{2!} \end{bmatrix} = N(\sigma)\,\frac{D^{(2)}(\sigma)}{2!} + D(\sigma)\,\frac{N^{(2)}(\sigma)}{2!} - N^{(1)}(\sigma)D^{(1)}(\sigma) \tag{17}$$

$$R_3(\sigma) = \begin{bmatrix} N(\sigma) & -N^{(1)}(\sigma) & \dfrac{N^{(2)}(\sigma)}{2!} & \dfrac{-N^{(3)}(\sigma)}{3!} \\ \\ D(\sigma) & D^{(1)}(\sigma) & \dfrac{D^{(2)}(\sigma)}{2!} & \dfrac{D^{(3)}(\sigma)}{3!} \end{bmatrix}$$

$$= N(\sigma)\,\frac{D^{(3)}(\sigma)}{3!} - D(\sigma)\,\frac{N^{(3)}(\sigma)}{3!} - N^{(1)}(\sigma)\,\frac{D^{(2)}(\sigma)}{2!} + D^{(1)}(\sigma)\,\frac{N^{(2)}(\sigma)}{2!}\,. \tag{18}$$

The coefficients $D_0(\sigma)$, $D_2(\sigma)$, $D_4(\sigma)$ are the only coefficients needed in the gain equations, and they are obtained in an analogous manner as shown.

$$D_0(\sigma) = \begin{bmatrix} D(\sigma) \\ \\ D(\sigma) \end{bmatrix} = (D(\sigma))^2 \tag{19}$$

$$D_2(\sigma) = \begin{bmatrix} D(\sigma) & -D^{(1)}(\sigma) & \dfrac{D^{(2)}(\sigma)}{2!} \\ \\ D(\sigma) & D^{(1)}(\sigma) & \dfrac{D^{(2)}(\sigma)}{2!} \end{bmatrix} = 2D(\sigma)\,\frac{D^{(2)}(\sigma)}{2!} - (D^{(1)}(\sigma))^2 \tag{20}$$

$$D_4(\sigma) = \begin{bmatrix} D(\sigma) & -D^{(1)}(\sigma) & \dfrac{D^{(2)}(\sigma)}{2!} & \dfrac{-D^{(3)}(\sigma)}{3!} & \dfrac{D^{(4)}(\sigma)}{4!} \\ \\ D(\sigma) & D^{(1)}(\sigma) & \dfrac{D^{(2)}(\sigma)}{2!} & \dfrac{D^{(3)}(\sigma)}{3!} & \dfrac{D^{(4)}(\sigma)}{4!} \end{bmatrix}$$

$$= 2D(\sigma)\,\frac{D^{(4)}(\sigma)}{4!} - 2D^{(1)}(\sigma)\,\frac{D^{(3)}(\sigma)}{3!} + \left(\frac{D^{(2)}(\sigma)}{2!}\right)^2. \tag{21}$$

The algorithm just given is utilized in a running example which will be used to illustrate further properties.

Example 1

The open loop transfer function is

$$KG(s) = \frac{K(s+4)}{s^4 + 16s^3 + 108s^2 + 400s + 800}$$

and the root locus and gain equations are desired.

$$N(\sigma) = \sigma + 4$$

and

$$D(\sigma) = \sigma^4 + 16\sigma^3 + 108\sigma^2 + 400\sigma + 800.$$

By differentiating the numerator and denominator successively, the root locus array is formed.

$$
\begin{array}{c|c|c|c|c|c}
R_0(\sigma) & R_1(\sigma) & R_2(\sigma) & R_3(\sigma) & R_4(\sigma) & R_5(\sigma) \\
\sigma + 4 & -1 & 0 & 0 & 0 & 0 \\
\sigma^4 + 16\sigma^3 + 108\sigma^2 + 400\sigma + 800 & 4\sigma^3 + 48\sigma^2 + 216\sigma + 400 & \dfrac{12\sigma^2 + 96\sigma + 216}{2!} & \dfrac{24\sigma + 96}{3!} & \dfrac{24}{4!} & 0
\end{array}.
$$

Using the rules for forming the cross products developed in the algorithm, the coefficients $R_0(\sigma)$, $R_1(\sigma)$, $R_2(\sigma)$, $R_3(\sigma)$, $R_4(\sigma)$ and $R_5(\sigma)$ can be written by inspection.

$$
\begin{aligned}
R_0(\sigma) &= (\sigma + 4)(\sigma^4 + 16\sigma^3 + 108\sigma^2 + 400^{\sigma} + 800) \\
R_1(\sigma) &= (\sigma + 4)(4\sigma^3 + 48\sigma^2 + 216\sigma + 400) \\
&\quad - (\sigma^4 + 16\sigma^3 + 108\sigma^2 + 400\sigma + 800) \\
R_2(\sigma) &= (\sigma + 4)(6\sigma^2 + 48\sigma + 108) \\
&\quad - (4\sigma^3 + 48\sigma^2 + 216\sigma + 400) \\
R_3(\sigma) &= (\sigma + 4)(4\sigma + 16) - (6\sigma^2 + 48\sigma + 108) \\
R_4(\sigma) &= (\sigma + 4) - (4\sigma + 16). \\
R_5(\sigma) &= -1.
\end{aligned}
$$

The following array is formed to obtain the coefficients $D_0(\sigma)$, $D_2(\sigma) \cdots$, $D_8(\sigma)$

$$
\begin{array}{c|c|c|c|c}
D_0(\sigma) & D_2(\sigma) & D_4(\sigma) & D_6(\sigma) & D_8(\sigma) \\
\sigma^4 + 16\sigma^3 + 108\sigma^2 + 400\sigma + 800 & -(4\sigma^3 + 48\sigma^2 + 216\sigma + 400),\ 6\sigma^2 + 48\sigma + 108 & -(4\sigma + 16) \quad 1 & 0 \quad 0 & 0 \quad 0 \\
\sigma^4 + 16\sigma^3 + 108\sigma + 400\sigma + 800 & 4\sigma^3 + 48\sigma^2 + 216\sigma + 400,\ 6\sigma^2 + 48\sigma + 108 & 4\sigma + 16 \quad 1 & 0 \quad 0 & 0 \quad 0
\end{array}
$$

$$
\begin{aligned}
D_0(\sigma) &= (\sigma^4 + 16\sigma^3 + 108\sigma^2 + 400\sigma + 800)^2 \\
D_2(\sigma) &= 2(6\sigma^2 + 48\sigma + 108)(\sigma^4 + 16\sigma^3 + 108\sigma^2 + 400\sigma + 800) - (4\sigma^3 + 48\sigma^2 + 216\sigma + 400)^2 \\
D_4(\sigma) &= 2(\sigma^4 + 16\sigma^3 + 108\sigma^2 + 400\sigma + 800) - 2(4\sigma + 16)(4\sigma^3 + 48\sigma^2 + 216\sigma + 400) + (6\sigma^2 + 48\sigma + 108)^2 \\
D_6(\sigma) &= 2(6\sigma^2 + 48\sigma + 108) - (4\sigma + 16)^2 \\
D_8(\sigma) &= 1.
\end{aligned}
$$

The root locus and gain equations are given by

$$
R_1(\sigma) - \omega^2 R_3(\sigma) - \omega^4 = 0
$$

$$
\frac{R_0(\sigma) - \omega^2 R_2(\sigma) + \omega^4 R_4(\sigma)}{D_0(\sigma) - \omega^2 D_2(\sigma) + \omega^4 D_6(\sigma) + \omega^8 D_8(\sigma)} = -\frac{1}{K}.
$$

IV. Breakaway Points from the Real Axis

The breakaway points are given by the real roots of (14) where $R_1(\sigma)$ is the first cross product of the root locus array, and can be conveniently given in the form of a determinant

$$
R_1(\sigma) = \begin{vmatrix} N(\sigma) & N^{(1)}(\sigma) \\ D(\sigma) & D^{(1)}(\sigma) \end{vmatrix} = 0. \tag{22}
$$

The angle at which the locus leaves the real axis will now be investigated. Differentiating (13) with respect to σ and ω, an expression for the derivative $d\omega/d\sigma$ is obtained as

$$
\frac{d\omega}{d\sigma} = \frac{R_1^{(1)}(\sigma) - \omega^2 R_3^{(1)}(\sigma) + \omega^4 R_5^{(1)}(\sigma) \pm \cdots}{2\omega\{R_3(\sigma) - 2\omega^2 R_5(\sigma) + 3\omega^4 R_7(\sigma) \pm \cdots\}}. \tag{23}
$$

Substituting $\omega = 0$ in (23), the angle at which the locus leaves the breakaway point is seen to be $\pi/2$ provided $R'_1(\sigma) \neq 0$ at the breakaway point σ_1. The fact that $dR_1(\sigma)/d\sigma$ at $\sigma = \sigma_1$ is equal to zero indicates a point of inflexion on the real axis and the equation $R_1(\sigma) = 0$ consists of multiple poles of order r ($r > 1$) at $\sigma = \sigma_1$ in addition to other poles along the real axis. If $R_1(\sigma)$ consists of only multiple poles at $\sigma = \sigma_1$, then the poles of the open loop transfer function lie symmetrically on the asymptotes and their loci are straight lines with the point $\sigma = \sigma_1$ being the asymptotic center. The following examples will illustrate these enunciated points.

Example 2

The breakaway points of the root locus of $G(s)$ of Example 1 are given by the real roots of the equation $R_1(\sigma) = 0$, i.e.,

$$
R_1(\sigma) = 3\sigma^4 + 48\sigma^3 + 300\sigma^2 + 864\sigma + 800 = 0.
$$

By synthetic division, the real roots are found to be at -6.35 and -1.65. There is, however, an easier method of solving this particular equation, i.e., by shifting the $j\omega$-axis.

Example 2

If

$$
G(s) = \frac{K}{s^4 + 12s^3 + 54s^2 + 108s + 145}
$$

the breakaway points are given by the real roots of the equation

$$
D^{(1)}(\sigma) = 0,
$$

i.e., $\sigma^3 + 9\sigma^2 + 27\sigma + 27 = 0$ which is equal to $(\sigma + 3)^3$. In this case all higher derivatives of $D^{(1)}(\sigma) = 0$ at $\sigma = -3$ vanish, and the point $\sigma = -3$ is indeed the asymptotic center.

V. Simplifications by the Shift of the Imaginary Axis

Since the root locus is an algebraic curve plotted for a given ratio of two polynomials, an amount of simplification will result if the $j\omega$-axis of the s-plane is shifted to a more desirable location. In the case of a stable open loop transfer function, the $j\omega$-axis will have to be shifted to the left. The resulting transfer function expressed in the shifted s-plane will be undoubtedly unstable, but after the root locus has been plotted, the $j\omega$-axis can be shifted back to the original location. This is one of the advantages in dealing with algebraic expressions which can be simplified by a proper transformation of axes. The simplification achieved will be quite striking in cases where a vertical axis of symmetry can be found. Usually this vertical axis of symmetry will pass through the asymptotic center given by

$$-\frac{(b_{n-1} - a_{m-1})}{n - m}.$$

This simplification will be illustrated by means of the open loop transfer function of Example 1.

Example 4

The asymptotic center for the open loop transfer function $G(s)$ of Example 1 is at $\sigma = -4$. A transformation $p = s + 4$ is applied to $G(s)$, resulting in the open loop transfer function in the p-plane

$$G(p) = \frac{Kp}{p^4 + 12p^2 + 48p + 160}$$

which is obviously unstable in the p-plane due to the absence of p^3 term. Using the root locus array, the root locus equation is given by

$$3\xi^4 + 12\xi^2 - 160 + 2(\xi^2 + 6)\omega^2 - \omega^4 = 0 \qquad p = \xi + j\omega.$$

This is a simpler expression to plot than the one obtained in Example 1. The breakaway points on the p-plane are obtained by solving the quadratic in ξ^2 viz., $3\xi^4 + 12\xi^2 - 160 = 0$ which is almost trivial when compared to solving the quartic $3\sigma^4 + 48\sigma^3 + 300\sigma^2 + 864\sigma + 800 = 0$. The roots of the equation $3\xi^4 + 12\xi^2 - 160 = 0$ are $\xi = \pm j3.09$, ± 2.35 which when the reverse transformation $\sigma = \xi - 4$ is applied, gives the roots to the original quartic $3\sigma^4 + 48\sigma^3 + 300\sigma^2 + 864\sigma + 800 = 0$ as $\sigma = -4 \pm j3.09$, -6.35, -1.65 and the breakaway points are the real roots -6.35 and -1.65 which agrees with Example 2.

VI. Intersection with the Imaginary Axis

The root locus array developed in Section III can be directly utilized to find the points of intersection on the imaginary axis. The value $\sigma = 0$, the equation of the imaginary axis, is substituted in the root locus array and the resulting expression,

$$R_1(0) - \omega^2 R_3(0) + \omega^4 R_5(0) \pm \cdots = 0 \qquad (24)$$

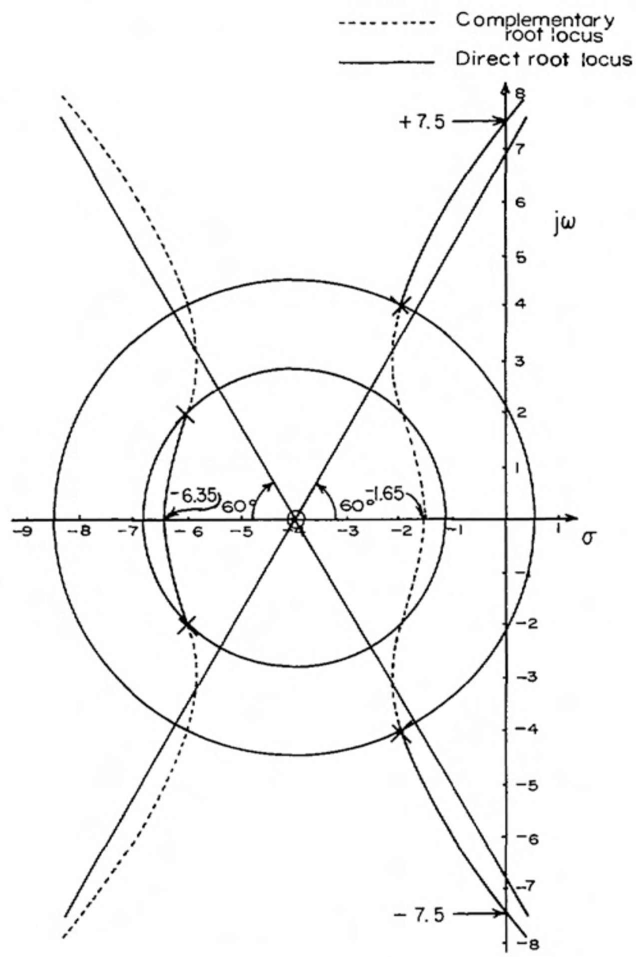

Fig. 1. Root locus of

$$KG(s) = \frac{K(s+4)}{s^4 + 16s^3 + 108s^2 + 400s + 800}.$$

is solved for the points on the imaginary axis. The open loop transfer function of Example 1, will be used to find the points of intersection of the locus and the imaginary axis.

Example 5

The value $\sigma = 0$ is substituted in the root locus array of Example 1 to obtain the following array

$$\begin{matrix} R_1(0) & & R_3(0) & & R_5(0) \\ \begin{bmatrix} 4 & -1 \\ 800 & 400 \end{bmatrix} & & \begin{bmatrix} 0 & 0 \\ 108 & 16 \end{bmatrix} & & \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \end{matrix}.$$

Using the cross product rule, the coefficients $R_1(0)$, $R_3(0)$ and $R_5(0)$ are 800, -44, and -1, respectively. Substituting these values in (24) and solving for the real roots, the points of intersection on the imaginary axis are $\omega = \pm 7.5$. The complete root locus is plotted in Fig. 1.

VII. Root Locus for Variable Parameters Other than the Gain K

If *one* of the real axis roots varies with the gain K kept at any constant value, the same techniques for plotting the locus can be applied in this case by suitably deriving a modified transfer function $G'(s)$. The real axis root $-\tau_1$, can be factored out of (3) as is now shown,

$$KG(s) = \frac{K(s^m + a_{m-1}s^{m-1} + \cdots + a_0)}{s^n + b_{n-1}'s^{n-1} + \cdots + b_1's + \tau_1(s^{n-1} + b_{n-2}''s^{n-2} + \cdots + b_0'')} \tag{25}$$

(it being assumed that root $= -\tau_1$ is not a multiple root). After some manipulation (25) can be written as

$$\frac{s^{n-1} + b_{n-2}''s^{n-2} + \cdots + b_0''}{(s^n + b_{n-1}'s^{n-1} + \cdots + b_1's) + K(s^m + a_{m-1}s^{m-1} + \cdots + a_0)} = -\frac{1}{\tau_1}. \tag{26}$$

A modified transfer function $G'(s)$ is defined as the ratio of the polynomials $N'(s)$ and $D'(s)$ where

$$N'(s) = s^{n-1} + b_{n-2}''s^{n-2} + \cdots + b_0'' \tag{27}$$

and

$$D'(s) = s^n + b_{n-1}'s^{n-1} + \cdots + b_1's + K(s^m + a_{m-1}s^{m-1} + \cdots + a_0). \tag{28}$$

The root locus and the gain equations are formulated by obtaining the arrays developed in the previous sections to the modified transfer function $N'(s)/D'(s)$. This technique fails in the case of complex conjugate poles and multiple poles on the real axis because *two* roots vary at the same time with the consequent result of nonlinear terms involving the roots. An example is provided where a real axis pole varies and the root locus is plotted.

Example 6

The open loop transfer function of a system is given by

$$KG(s) = \frac{K}{(s+2)(s-a)}.$$

The root a varies over a wide range. The root locus plot is desired with K as a constant parameter.

After factoring the root a, the open loop transfer function can be modified to give the following transfer function corresponding to (26).

$$G'(s) = \frac{s+2}{s^2 + 2s + K}.$$

The root locus array is formed

$$\begin{matrix} R_0(\sigma) & R_1(\sigma) & R_2(\sigma) & R_3(\sigma) \\ \left.\begin{matrix} \sigma + 2 \\ \sigma^2 + 2\sigma + K \end{matrix}\right] & \left.\begin{matrix} -1 \\ 2\sigma + 2 \end{matrix}\right] & \left.\begin{matrix} 0 \\ 1 \end{matrix}\right] & \left.\begin{matrix} 0 \\ 0 \end{matrix}\right] \end{matrix}$$

from which the root locus equation is the circle

$$\omega^2 + (\sigma + 2)^2 = (\sqrt{K})^2$$

with center at $\sigma = -2$ and radius $\sqrt{K}$.

To find the corresponding value for the root position a, the following array is formed

$$\begin{matrix} D_0(\sigma) & D_2(\sigma) & D_4(\sigma) \\ \left.\begin{matrix} \sigma^2 + 2\sigma + K \\ \sigma^2 + 2\sigma + K \end{matrix}\right] & \left.\begin{matrix} -2(\sigma+1) & 1 \\ 2(\sigma+1) & 1 \end{matrix}\right] & \left.\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}\right] \end{matrix}$$

and the gain equation is written as

$$\frac{(\sigma + 2)(\sigma^2 + 2\sigma + K) - (\sigma + 2)\omega^2}{(\sigma^2 + 2\sigma + K)^2 + 2(\sigma^2 + 2\sigma + K)\omega^2 + \omega^4} = +\frac{1}{a}$$

from which values of a can be found from any given ω, σ, and the constant parameter K. In particular, the values of a corresponding to the breakaway points on the real axis are obtained by substituting these values in the equation

$$a = \frac{+(\sigma^2 + 2\sigma + K)}{\sigma + 2}.$$

The root locus is plotted in Fig. 2.

Discussion

For any given transfer function $G(s)$ expressed as a ratio of polynomials, arrays are developed from which the root locus equation and the gain equation can be written readily. The root locus equation is simpler than the gain equation, and it is of the form

$$\omega\{R_1(\sigma) - \omega^2 R_3(\sigma) + \omega^4 R_5(\sigma) \pm \cdots\} = 0.$$

By substituting $\omega = 0$ in the root locus array, an equation for finding the breakaway points is obtained. Having obtained the breakaway points, suitable values of σ starting close to the breakaway point are substituted into the root locus equation to obtain solutions for ω^2 which are real and positive. The values of σ and ω are then substituted in the gain array to find the corresponding value of K. The method suggested before is particularly useful for lower order systems (i.e., the sum of the degrees of the numerator and denominator of $G(s)$ is less than 7).

Any given $G(s)$ can be transformed into a more manageable form by a shift of the imaginary axis to the left of the s-plane. If a vertical axis of symmetry for the root locus is known to exist then the imaginary axis is shifted

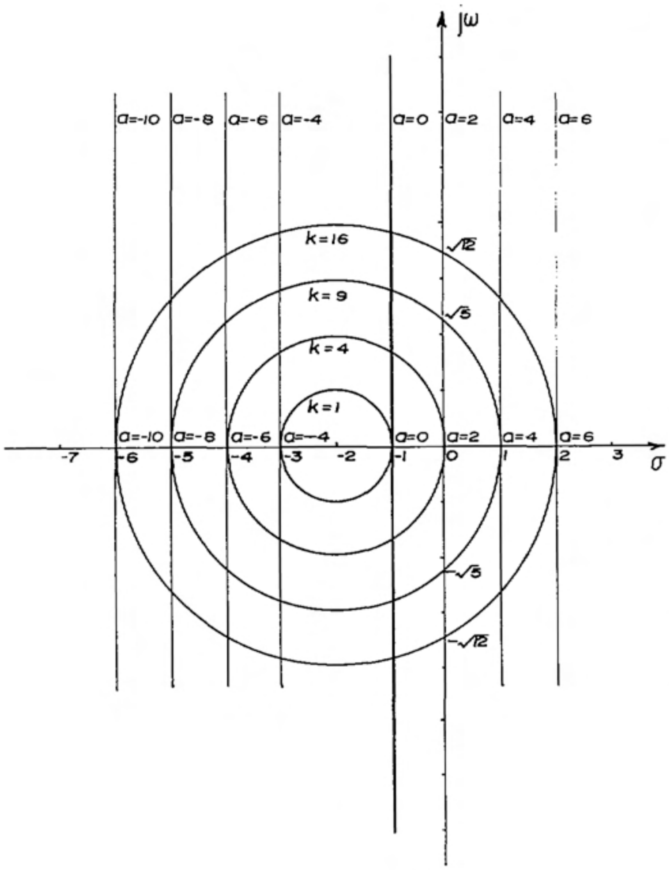

Fig. 2. Root locus of

$$KG(s) = \frac{K}{(s+2)(s+a)}.$$

to the asymptotic center through which the vertical axis of symmetry will pass. In the case where there is no vertical axis of symmetry then the optimum shift of the imaginary axis for ease of computation will require further study.

The usefulness of these arrays is also demonstrated in the case where one of the roots of the transfer function varies.

References

[1] G. A. Bendrikov and K. F. Teodorchik, "The laws of root migration for third and fourth power linear algebraic equations as the free terms changes continuously," *Automat. remote Control*, vol. 16, no. 3, 1955.

[2] G. A. Bendrikov and K. F. Teodorchik, "The theory of derivative control in third order linear systems, *Automat. remote Control*, pp. 516–517, May 1957.

[3] G. A. Bendrikov and K. F. Teodorchik, "The analytic theory of constructing root loci," *Automat. remote Control*, pp. 340–344, March 1959.

[4] K. F. Teodorchik and G. A. Bendrikov, "The methods of plotting root paths of linear systems and for qualitative determination of path type," *Proc. of IFAC Congress*, vol. 1, pp. 8–12, 1960.

[5] W. R. Evans, "Control system synthesis by the root locus method," *AIEE Trans.*, vol. 69, pt. 1, pp. 66–69, 1950.

[6] J. A. Aseltine, "Feedback system synthesis by the inverse root-locus method," *IRE Nat'l. Conv. Rec.*, pt. 2, pp. 13–17, 1956.

[7] K. S. Narendra, "Inverse root-locus, reversed root-locus or complementary root-locus," *IRE Trans. on Automatic Control*, (*Correspondence*), pp. 359–360, September 1961.

[8] K. Steiglitz, "An analytical approach to root loci," *IRE Trans. on Automatic Control*, vol. AC-6, pp. 326–332, September 1961.

[9] C. K. Wojcik, "Analytical representation of root locus," *J. Basic Eng.*, pp. 37–43, March 1964.

[10] V. Krishnan, "Semi-analytic approach to root locus," Microwave Research Institute, Polytechnic Institute of Brooklyn, N. Y., Rept. R-452.27.65 to the Joint Services Technical Advisory Committee, pp. 105–111, March 1965.

[11] J. G. Truxal, *Automatic feedback control system synthesis*. New York: McGraw-Hill, 1955.

[12] B. P. Bhattacharyya, "Root locus equations of the fourth degree," *Internat'l J. of Control*, vol. 1, no. 6, pp. 533–556, 1965.